

CURVATURE OF HIGHER DIRECT IMAGE SHEAVES AND ITS APPLICATION ON NEGATIVE-CURVATURE CRITERION FOR THE WEIL-PETERSSON METRIC

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ABSTRACT. We shall show that q -semipositivity of the vector bundle E over a Kähler total space \mathcal{X} implies the Griffiths-semipositivity of the q -th direct image of $\mathcal{O}(K_{\mathcal{X}/B} \otimes E)$. As an application, we shall give a negative-curvature criterion for the generalized Weil-Petersson metric on the base manifold.

KEYWORDS: Higher direct image, Hodge theory, Chern curvature, $\bar{\partial}$ -equation, Weil-Petersson metric, canonically polarized manifold, Calabi-Yau manifold.

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1. INTRODUCTION

1.1. Set up. Let $\pi : \mathcal{X} \rightarrow B$ be a proper holomorphic submersion from a complex manifold \mathcal{X} to a connected complex manifold B . We call B the base manifold of the fibration π . Assume that B is m -dimensional and each fibre $X_t := \pi^{-1}(t)$ of π is n -dimensional. Let E

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be a holomorphic vector bundle over \mathcal{X} . Denote by E_t the restriction of E to X_t . Let us denote by $K_{\mathcal{X}/B}$ the relative canonical bundle on \mathcal{X} . Recall that

$$(1.1) \quad K_{\mathcal{X}/B} = K_{\mathcal{X}} - \pi^*(K_B).$$

Let us denote by $H^{p,q}(E_t)$ the space of E_t -valued $\bar{\partial}$ -closed (p, q) -type Dolbeault cohomology classes over X_t . Let us consider

$$(1.2) \quad \mathcal{H}^{p,q} := \{H^{p,q}(E_t)\}_{t \in B}.$$

It is known that $\mathcal{H}^{p,q}$ has a natural holomorphic vector bundle structure if

A: *The dimension of $H^{p,q}(E_t)$ does not depend on $t \in B$.*

The usual way to prove this fact is to use the base change theorem. In fact, by the base change theorem, **A** implies that the q -th direct image sheaf

$$(1.3) \quad \underline{E}^{p,q} := R^q \pi_* \mathcal{O}(\wedge^p T_{\mathcal{X}/B}^* \otimes E).$$

is locally free and the fibres of the holomorphic vector bundle associated to $\underline{E}^{p,q}$ are

$$(1.4) \quad H^q(X_t, \mathcal{O}(\wedge^p T_{X_t}^* \otimes E_t)), \quad t \in B.$$

By Dolbeault's theorem,

$$(1.5) \quad H^q(X_t, \mathcal{O}(\wedge^p T_{X_t}^* \otimes E_t)) \simeq H^{p,q}(E_t),$$

thus there is a holomorphic vector bundle structure on $\mathcal{H}^{p,q}$.

In this paper, we shall use the Newlander-Nirenberg theorem [24] and a theorem of Kodaira-Spencer (see Page 347 in [16]) to construct a holomorphic vector bundle structure on $\mathcal{H}^{p,q}$ directly (see Theorem 3.1 below).

Assume that \mathcal{X} possesses a Kähler form ω , put

$$(1.6) \quad \omega^t := \omega|_{X_t} > 0, \text{ on } X_t.$$

Let h be a Hermitian metric on E . By the Hodge theory, every cohomology class in $H^{p,q}(E_t)$ has a unique harmonic representative in

$$(1.7) \quad \mathcal{H}_t^{p,q} := \ker \bar{\partial}^t \cap \ker (\bar{\partial}^t)^*,$$

where $\bar{\partial}^t := \bar{\partial}|_{X_t}$ and $(\bar{\partial}^t)^*$ is the adjoint of $\bar{\partial}^t$ with respect to ω^t and h . In this paper, we shall identify a cohomology class in $H^{p,q}(E_t)$ with its harmonic representative in $\mathcal{H}_t^{p,q}$. Thus we have

$$(1.8) \quad \mathcal{H}^{p,q} = \{\mathcal{H}_t^{p,q}\}_{t \in B}.$$

The natural L^2 -inner product on each harmonic space $\mathcal{H}_t^{p,q}$ defines a Hermitian metric on $\mathcal{H}^{p,q}$. Let us denote by D the associated Chern connection on $\mathcal{H}^{p,q}$. Put

$$(1.9) \quad D = \sum dt^j \otimes D_{t^j} + d\bar{t}^{\bar{j}} \otimes \bar{D}_{t^{\bar{j}}}.$$

Then

$$(1.10) \quad D^2 = \sum dt^j \wedge d\bar{t}^{\bar{k}} \otimes \Theta_{j\bar{k}}, \quad \Theta_{j\bar{k}} := D_{t^j} \bar{D}_{t^{\bar{k}}} - \bar{D}_{t^{\bar{k}}} D_{t^j}.$$

Denote by (\cdot, \cdot) and $\|\cdot\|$ the associated inner product and norm on the fibre, $\mathcal{H}_t^{p,q}$, of $\mathcal{H}^{p,q}$ respectively. By definition, $\mathcal{H}^{p,q}$ is semi-positive in the sense of Griffiths if and only if

$$\sum (\Theta_{j\bar{k}} u, u) \xi^j \bar{\xi}^k \geq 0, \quad \forall u \in \mathcal{H}_t^{p,q}, \quad \xi \in \mathbb{C}^m.$$

Moreover, $\mathcal{H}^{p,q}$ is semi-positive in the sense of Nakano if and only if

$$\sum (\Theta_{j\bar{k}} u_j, u_k) \geq 0, \quad \forall u_j \in \mathcal{H}_t^{p,q}, \quad 1 \leq j \leq m.$$

In this paper, we shall give a curvature formula for $\mathcal{H}^{p,q}$ based on the formulas in [3, 4, 7] for $\mathcal{H}^{n,0}$, see [18] and [19] for earlier results and [29], [25], [17], [14], [20] and [21] for other generalizations and related results.

1.2. Main result. Denote by $\Theta(E, h)$ (resp. $\Theta(E_t, h)$) the curvature operator of the Chern connection

$$(1.11) \quad d^E := \bar{\partial} + \partial^E, \quad (\text{resp. } d^{E_t} := \bar{\partial}^t + \partial^{E_t}),$$

on (E, h) (resp. (E_t, h)) respectively. Put

$$(1.12) \quad \omega_q := \frac{\omega^q}{q!}, \quad \omega_q^t := \frac{(\omega^t)^q}{q!}.$$

We shall use the following definition:

Definition 1.1. $\Theta(E, h)$ is said to be q -semipositive on \mathcal{X} with respect to ω if

$$(1.13) \quad \omega_q \wedge c_q \{i\Theta(E, h)u, u\} \geq 0, \quad \text{on } \mathcal{X},$$

for every E -valued $(n + m - q - 1, 0)$ -form u in \mathcal{X} . where $c_q = i^{(n+m-q-1)^2}$ is chosen such that

$$(1.14) \quad \omega_{q+1} \wedge c_q \{u, u\} \geq 0,$$

as a semi-positive volume form on the total space \mathcal{X} .

Our main theorem is the following:

Theorem 1.1. Assume that the total space is Kähler and the dimension of $H^{n,q}(E_t)$ does not depend on $t \in B$. Assume further that $\Theta(E, h)$ is q -semipositive with respect to a Kähler form on the total space. Then $\mathcal{H}^{n,q}$ is Griffiths-semipositive.

Remark: The following fact is also true:

With the assumptions in the above theorem, assume further that

$$(1.15) \quad \mathcal{H}_t^{n,q} = \ker \partial^{E_t} \cap \ker (\partial^{E_t})^*, \quad \forall t \in B,$$

then $\mathcal{H}^{n,q}$ is Nakano-semipositive.

If $q = 0$ then (1.15) is always true. If $q \geq 1$, by Siu's $\partial\bar{\partial}$ -Bochner formula (see [27] or [2]), we know that (1.15) is true in case

$$(1.16) \quad i\Theta(E_t, h) \wedge \omega_{q-1}^t \equiv 0,$$

on X_t for every $t \in B$.

1.3. Applications. We shall use our main theorem to study the curvature properties of the base manifold B . Let us denote by κ the Kodaira-Spencer mapping

$$(1.17) \quad \kappa : v \mapsto \kappa(v) \in H^{0,1}(T_{X_t}) \simeq H^{n,n-1}(T_{X_t}^*)^*, \quad \forall v \in T_t B, \quad t \in B.$$

We shall introduce the following definition:

Definition 1.2. We call the pull back pseudo-metric on T_B defined by

$$(1.18) \quad \|v\|_{WP} := \|\kappa(v)\|_{H^{n,n-1}(T_{X_t}^*)^*}, \quad \forall v \in T_t B$$

the generalized Weil-Petersson metric on B .

Assume that the dimension of $H^{0,1}(T_{X_t})$ does not depend on t in B . Then we know that the dimension of the dual space, $H^{n,n-1}(T_{X_t}^*)$, of $H^{0,1}(T_{X_t})$ does not depend on t in B . Moreover, we shall prove that (see Proposition 4.1)

$$(1.19) \quad \kappa : v \mapsto \kappa(v) \in H^{n,n-1}(T_{X_t}^*)^*, \forall v \in T_t B,$$

defines a holomorphic bundle map from T_B to the **dual** bundle of

$$(1.20) \quad \mathcal{H}^{n,n-1} := \{H^{n,n-1}(T_{X_t}^*)\}_{t \in B},$$

Assume that the total space \mathcal{X} possesses a Kähler form ω . We shall introduce the following definition:

Definition 1.3. *The relative cotangent bundle $T_{\mathcal{X}/B}^*$ is said to be $(n-1)$ -semipositive with respect to ω if there is a smooth metric, say h , on the relative cotangent bundle such that $\Theta(T_{\mathcal{X}/B}^*, h)$ is $(n-1)$ -semipositive with respect to ω .*

By Theorem 1.1, if the relative cotangent bundle is $(n-1)$ -semipositive with respect to ω then $\mathcal{H}^{n,n-1}$ is Griffiths-semipositive. Assume further that κ is injective. Then T_B is Griffiths-seminegative with respect to the generalized Weil-Petersson metric. Inspired by [23] and [9], we shall introduce the following definition:

Definition 1.4. $\|\cdot\|_{WP}$ defines a Griffiths-seminegative singular metric if and only if for every holomorphic vector field $v : t \mapsto v^t$ on the base, $\log \|v\|_{WP}$ is plurisubharmonic or equal to $-\infty$ identically, where $\|v\|_{WP}$ denotes the upper semicontinuous regularization of $t \mapsto \|v^t\|_{WP}$.

We shall prove that:

Theorem 1.2. *Let π be a proper holomorphic submersion from a Kähler manifold (\mathcal{X}, ω) to a complex manifold B . Assume that the relative cotangent bundle is $(n-1)$ -semipositive with respect to ω . Then the associated generalized Weil-Petersson metric defines a Griffiths-seminegative singular metric on T_B .*

Remark A: If the canonical line bundle of each fibre is positive then by Aubin-Yau's theorem (see [1] and [34]), each fibre possesses a unique Kähler-Einstein metric, which defines a smooth Hermitian metric, say h , on the relative cotangent bundle. Then we know that the cotangent bundle of *each fibre* is $(n-1)$ -semipositive. Moreover, if $n = 1$ then it is well known that relative cotangent bundle is 0-semipositive. But for a general canonically polarized family, we don't know whether the relative cotangent bundle is $(n-1)$ -semipositive or not, for related results, say [25]. In [10], we shall introduce another way to study the curvature properties of the base manifold of a canonically polarized family.

Remark B: Given a Kähler total space (\mathcal{X}, ω) , if the canonical line bundle of each fibre is trivial then by Yau's theorem [34], we know that there is a smooth function, say ϕ , on \mathcal{X} such that $\omega + i\partial\bar{\partial}\phi$ is Ricci-flat on each fibre. Let us denote by h the smooth Hermitian metric on the relative cotangent bundle defined by $\omega + i\partial\bar{\partial}\phi$. Then we know that the cotangent bundle of *each fibre* is $(n-1)$ -semipositive. But we don't know whether the relative cotangent bundle is $(n-1)$ -semipositive or not, except for some special case, e.g. deformation of torus or other families with flat relative cotangent bundle. In [33], we shall introduce another metric to study the base manifold of a Calabi-Yau family.

1.4. List of notations.

Basic notions:

1. $\pi : \mathcal{X} \rightarrow B$ is a proper holomorphic submersion, E : holomorphic vector bundle on \mathcal{X} ;
2. $X_t := \pi^{-1}(t)$ is the fibre at t , $E_t := E|_{X_t}$;
3. $d^E := \bar{\partial} + \partial^E$ is the Chern connection on E , $d^{E_t} := \bar{\partial}^t + \partial^{E_t}$ is its restriction;
4. $\Theta(E, h) := (d^E)^2$ is Chern curvature of E , $\Theta(E_t, h) := (d^{E_t})^2$;
5. $H^{p,q}(E_t)$: Dolbeault cohomology group on X_t ;
6. $\mathcal{H}_t^{p,q} := \ker \bar{\partial}^t \cap \ker (\bar{\partial}^t)^* \simeq H^{p,q}(E_t)$ is the harmonic space;
7. $\mathcal{H}^{p,q} := \{\mathcal{H}_t^{p,q}\}_{t \in B} \simeq \{H^{p,q}(E_t)\}_{t \in B}$;
8. $D = \sum dt^j \otimes D_{tj} + d\bar{t}^j \otimes \bar{\partial}_{tj}$ is the Chern connection on $\mathcal{H}^{p,q}$;
9. $\Theta_{j\bar{k}} := D_{tj} \bar{\partial}_{t\bar{k}} - \bar{\partial}_{t\bar{k}} D_{tj}$ is the curvature operator on $\mathcal{H}^{p,q}$.

Other notations:

1. i_t : the inclusion mapping $X_t \hookrightarrow \mathcal{X}$;
2. t : local holomorphic coordinate system on B , t^j : components of t ;
3. $\delta_V := V \lrcorner$ means contraction of a form with a vector field V ;
4. V_j : smooth $(1,0)$ -vector field on \mathcal{X} such that $\pi_* V_j = \partial/\partial t^j$, L_{V_j} : usual Lie-derivative;
5. $L_j := d^E \delta_{V_j} + \delta_{V_j} d^E = [d^E, \delta_{V_j}]$, $L_{\bar{j}} := d^E \delta_{\bar{V}_j} + \delta_{\bar{V}_j} d^E = [d^E, \delta_{\bar{V}_j}]$;
6. $u : t \mapsto u^t \in \mathcal{H}_t^{p,q}$ is a section of $\mathcal{H}^{p,q}$;
7. $\Gamma(\mathcal{H}^{p,q})$: space of smooth sections of $\mathcal{H}^{p,q}$;
8. a smooth E -valued (p,q) -form \mathbf{u} on \mathcal{X} : a representative of $u \in \Gamma(\mathcal{H}^{p,q})$;
9. $*$: Hodge-Poincaré-de Rham star operator;
10. \mathbf{u}^* : dual-representative of u such that $\mathbf{u}^*|_{X_t} = *u^t$ for every $t \in B$;
11. \mathbb{H}^t : orthogonal projection to $\mathcal{H}_t^{p,q}$;
12. $\kappa : \partial/\partial t^j \mapsto \kappa(\partial/\partial t^j)$ is the Kodaira-Spencer mapping.

2. MOTIVATION: THE PRODUCT CASE

2.1. Griffiths-positivity of the bundle of harmonic forms. We shall give an example to show the ideas behind the proof of Theorem 1.1.

Let us consider the following product case: Let L be a holomorphic line bundle over a compact Kähler manifold (X, ω) . Let h be a fixed smooth metric on L and let ϕ be a smooth function on

$$(2.1) \quad \mathcal{X} := X \times \mathbb{B}.$$

Put

$$(2.2) \quad h^t = h e^{-\phi^t}, \quad \phi^t := \phi|_{X \times \{t\}}.$$

Thus $\{h^t\}_{t \in B}$ defines a smooth metric, say \tilde{h} , on $L \times \mathbb{B}$. We shall consider

$$(2.3) \quad \mathcal{H}^{n,q} := H^{n,q}(L) \times \mathbb{B} = \{\mathcal{H}_t^{n,q}\}_{t \in \mathbb{B}},$$

where \mathbb{B} is the unit ball in \mathbb{C}^m and each $H_t^{n,q}$ is the harmonic space with respect to ω and h^t . We know that $\mathcal{H}^{n,q}$ is a trivial vector bundle with non-trivial metric.

For every $u \in H^{n,q}(L)$ there is an associated holomorphic section

$$(2.4) \quad t \mapsto u^t := \mathbb{H}^t(\mathbf{u}),$$

where each \mathbf{u} is a fixed $\bar{\partial}_X$ -closed representative of u and \mathbb{H}^t is the orthogonal projection to the harmonic space $\mathcal{H}_t^{n,q}$. Now we know that

$$(2.5) \quad (u, v) : t \mapsto (u^t, v^t), \quad u, v \in H^{n,q}(L)$$

is a smooth function on \mathbb{B} . Moreover, we have

$$(2.6) \quad (u, v)_{j\bar{k}} = (D_{t^j} u^t, D_{t^k} v^t) - (\Theta_{j\bar{k}} u^t, v^t).$$

By definition, we have

$$(2.7) \quad (u^t, v^t) = \int_X \{u^t, *v^t\} e^{-\phi^t},$$

where $\{\cdot, \cdot\}$ is the pairing associated to h . Thus we have

$$(2.8) \quad (u, v)_j = (L_j u^t, v^t) + \int_X \{u^t, L_{\bar{j}} * v^t\} e^{-\phi^t},$$

where

$$(2.9) \quad L_j := \partial/\partial t^j - \phi_j \cdot, \quad L_{\bar{j}} := \partial/\partial \bar{t}^j.$$

Since $(u^t)^* \wedge \omega_q = i^{(n-q)^2} (-1)^{n-q} u^t$, we have

$$(2.10) \quad L_j * = * L_j, \quad L_{\bar{j}} * = * L_{\bar{j}}, \quad \forall 1 \leq j \leq m.$$

Now we have

$$(2.11) \quad (u, v)_j = (L_j u^t, v^t) + (u^t, L_{\bar{j}} v^t).$$

Let v be a fixed (does not depend on t) $\bar{\partial}_X$ -closed representative of v . Thus each $v^t - \mathbf{v}$ is $\bar{\partial}_X$ -exact, which implies that

$$(2.12) \quad L_{\bar{j}} v^t = L_{\bar{j}} (v^t - \mathbf{v}),$$

is $\bar{\partial}_X$ -exact. Then we have

$$(2.13) \quad (u^t, L_{\bar{j}} v^t) \equiv 0.$$

Hence we get that

$$(2.14) \quad D_{t^j} u^t = \mathbb{H}^t(L_j u^t),$$

and

$$(2.15) \quad (u, v)_{j\bar{k}} = (L_j u^t, v^t)_{\bar{k}} = (L_{\bar{k}} L_j u^t, v^t) + (L_j u^t, L_{\bar{k}} v^t).$$

Notice that

$$(2.16) \quad L_{\bar{k}} L_j u^t = L_j L_{\bar{k}} u^t - \phi_{j\bar{k}} u^t.$$

Then we have

$$(2.17) \quad (L_{\bar{k}} L_j u^t, v^t) = -(\phi_{j\bar{k}} u^t, v^t) + (L_{\bar{k}} u^t, v^t)_j - (L_{\bar{k}} u^t, L_{\bar{j}} v^t).$$

Since $(L_{\bar{k}} u^t, v^t) \equiv 0$, thus we have

$$(2.18) \quad (u, v)_{j\bar{k}} = -(\phi_{j\bar{k}} u^t, v^t) - (L_{\bar{k}} u^t, L_{\bar{j}} v^t) + (L_j u^t, L_{\bar{k}} v^t).$$

The following lemma is a crucial step (see Lemma 3.6 for the general case).

Lemma 2.1. *Put $a_j^t = D_{t^j} u^t - L_j u^t$, then each a_j^t is the L^2 -minimal solution of*

$$(2.19) \quad \bar{\partial}_X a_j^t = -\bar{\partial}_X L_j u^t = \bar{\partial}_X \phi_j \wedge u^t.$$

Proof. It suffices to show that $\bar{\partial}_X^*(a^t) \equiv 0$. Since

$$\bar{\partial}_X^* = - * (\partial_X - \partial_X \phi^t) *, \quad \bar{\partial}_X^*(a^t) = -\bar{\partial}_X^*(L_j u^t),$$

it is enough to show $*L_j u^t$ is $(\partial_X - \partial_X \phi^t)$ -closed. By (2.10), we have

$$(2.20) \quad *L_j u^t = L_j *u^t.$$

Thus

$$(2.21) \quad (\partial_X - \partial_X \phi^t) *L_j u^t = (\partial_X - \partial_X \phi^t)L_j *u^t.$$

Since each u^t is harmonic, thus

$$(2.22) \quad (\partial_X - \partial_X \phi^t) *u^t \equiv 0.$$

Thus it is enough to show that

$$(2.23) \quad [\partial_X - \partial_X \phi^t, L_j] \equiv 0,$$

which follows by direct computation. \square

Fix $u_1, \dots, u_m \in H^{n,q}(L)$, then we have

$$(2.24) \quad \sum (\Theta_{j\bar{k}} u_j^t, u_k^t) = -||a^t||^2 + \sum (\phi_{j\bar{k}} u_j^t, u_k^t) + \sum (L_{\bar{k}} u_j^t, L_{\bar{j}} u_k^t),$$

where

$$(2.25) \quad a^t := \sum a_j^t, \quad \bar{\partial}_X a^t = \sum \bar{\partial}_X \phi_j \wedge u_j^t := c^t$$

By the classical Bochner-Kodaira-Nakano formula, if

$$(2.26) \quad i^{(n-q-1)^2} \omega_q \wedge \{i\Theta(L, h^t)u, u\} > 0, \quad \text{on } X,$$

for every $(n-q-1, 0)$ -form u that has no zero point in X then we have

$$(2.27) \quad ||a^t||^2 \leq ([i\Theta(L, h^t), \Lambda_\omega]^{-1} c^t, c^t) = \sum (T_{j\bar{k}} u_j^t, u_k^t),$$

where Λ_ω is the adjoint of $\omega \wedge$ and

$$(2.28) \quad T_{j\bar{k}} := (\bar{\partial}_X \phi_k \wedge \cdot)^* [i\Theta(L, h^t), \Lambda_\omega]^{-1} (\bar{\partial}_X \phi_j \wedge \cdot)$$

Now we have

$$(2.29) \quad \sum (\Theta_{j\bar{k}} u_j^t, u_k^t) = R + \sum ((\phi_{j\bar{k}} - T_{j\bar{k}}) u_j^t, u_k^t) + \sum (L_{\bar{k}} u_j^t, L_{\bar{j}} u_k^t),$$

where

$$(2.30) \quad R := ([i\Theta(L, h^t), \Lambda_\omega]^{-1} c^t, c^t) - ||a^t||^2 \geq 0.$$

We shall use the following lemma (see Lemma 3.10 for the general case):

Lemma 2.2. *Assume that (2.26) is true for every $t \in \mathbb{B}$. Then we have*

$$(2.31) \quad \sum ((\phi_{j\bar{k}} - T_{j\bar{k}}) u_j^t, u_k^t) \wedge i^{m^2} dt \wedge \bar{d}t = \pi_* \left(i^{(m+n-1-q)^2} \omega_q \wedge \{i\Theta(L \times \mathbb{B}, \tilde{h}) \mathbf{u}^*, \mathbf{u}^*\} \right),$$

where $\mathbf{u}^* := \mathbf{u}_j^* \wedge (\partial/\partial t^j \lrcorner dt)$ and each \mathbf{u}_j^* satisfies that

$$(2.32) \quad i_t^* \left(\partial/\partial t^j \lrcorner (\omega_q \wedge \Theta(L \times \mathbb{B}, \tilde{h}) \mathbf{u}_j^*) \right) \equiv 0, \quad i_t^* \mathbf{u}_j^* = *u_j^t,$$

on X for every $t \in \mathbb{B}$.

Thus we get the following result (see Theorem 1.1 for the general case):

Theorem 2.3. *Assume that*

$$(2.33) \quad \omega_q \wedge i\Theta(L \times \mathbb{B}, \tilde{h}) \geq 0, \quad \text{on } \mathcal{X},$$

and

$$(2.34) \quad \omega_q \wedge i\Theta(L, h^t) > 0, \quad \text{on } X, \text{ for all } t \in \mathbb{B}.$$

Then we have

$$(2.35) \quad \sum (\Theta_{j\bar{k}} u_j^t, u_k^t) \geq \sum (L_{\bar{k}} u_j^t, L_{\bar{j}} u_k^t).$$

In particular, $\mathcal{H}^{n,q}$ is Griffiths-semipositive.

We shall show in the next section that $\mathcal{H}^{n,q}$ can be seen as a holomorphic quotient bundle of a Nakano-semipositive bundle. But in general, a holomorphic quotient bundle of a Nakano-semipositive bundle is not Nakano-semipositive (see Page 340 in [13] for a counterexample).

2.2. Nakano-positivity of the bundle of $\bar{\partial}$ -closed forms. Let us denote by $\ker \bar{\partial}$ (resp. $\text{Im } \bar{\partial}$) the space of smooth $\bar{\partial}$ -closed (resp. $\bar{\partial}$ -exact) L -valued (n, q) -forms on X respectively. Then we have the following trivial bundles:

$$(2.36) \quad \mathcal{K} := \ker \bar{\partial} \times \mathbb{B}, \quad \mathcal{I} := \text{Im } \bar{\partial} \times \mathbb{B}.$$

But in general the metrics on \mathcal{K} and \mathcal{I} defined by $\{h^t\}_{t \in \mathbb{B}}$ are not trivial. By definition, we know that $\mathcal{H}^{n,q}$ is just the quotient bundle \mathcal{K}/\mathcal{I} . And the metric on $\mathcal{H}^{n,q}$ is just the quotient metric (see [32] for more results).

For every $u, v \in \ker \bar{\partial}$, we shall write

$$(2.37) \quad (u, v) : t \mapsto \int_X \{u, *v\} e^{-\phi^t}.$$

Let us denote by $\Theta_{j\bar{k}}^{\mathcal{K}}$ the curvature operators on \mathcal{K} . Fix $u_1, \dots, u_m \in \ker \bar{\partial}$. Since now

$$(2.38) \quad L_{\bar{j}} u_k \equiv 0, \quad \forall 1 \leq j, k \leq m,$$

we know that the following theorem is true.

Theorem 2.4. *With the assumptions in Theorem 2.3, then we have*

$$(2.39) \quad \sum (\Theta_{j\bar{k}}^{\mathcal{K}} u_j, u_k) = R + \sum ((\phi_{j\bar{k}} - T_{j\bar{k}}) u_j, u_k) \geq 0.$$

In particular, \mathcal{K} is Nakano-semipositive.

Remark: One may also study the positivity properties of the bundle of $\bar{\partial}$ -closed forms for non-trivial fibrations (see [31]).

3. CURVATURE FORMULA

We shall give a curvature formula for $\mathcal{H}^{p,q}$ in this section.

3.1. Holomorphic vector bundle structure on $\mathcal{H}^{p,q}$. By a theorem of Kodaira-Spencer (see Page 349 in [16]), we know that $\mathcal{H}^{p,q}$ has a smooth complex vector bundle structure if **A** is true. More precisely, **A** implies that for every $t_0 \in B$ and every $u^{t_0} \in \mathcal{H}_{t_0}^{p,q}$ (see (1.7) for the definition of the $\bar{\partial}$ -harmonic space $\mathcal{H}_{t_0}^{p,q}$), there is a smooth E -valued (p, q) -form, say \mathbf{u} , on \mathcal{X} such that

$$(3.1) \quad \mathbf{u}|_{X_{t_0}} = u^{t_0},$$

and

$$(3.2) \quad \mathbf{u}|_{X_t} \in \mathcal{H}_t^{p,q},$$

for every $t \in B$. Then the smooth vector bundle structure $\mathcal{H}^{p,q}$ can be defined as follows:

Definition 3.1. We call $u : t \mapsto u^t \in \mathcal{H}_t^{p,q}$ a smooth section of $\mathcal{H}^{p,q}$ if there exists a smooth E -valued (p, q) -form, say \mathbf{u} , on \mathcal{X} such that $\mathbf{u}|_{X_t} = u^t$, $\forall t \in B$. And we call \mathbf{u} a representative of u . We shall denote by $\Gamma(\mathcal{H}^{p,q})$ the space of smooth sections of $\mathcal{H}^{p,q}$.

By using the Newlander-Nirenberg theorem, we shall prove that:

Theorem 3.1. Assume that $\mathcal{H}^{p,q}$ satisfies **A**. Then $D^{0,1} := \sum d\bar{t}^j \otimes \bar{\partial}_{tj}$ defines a holomorphic vector bundle structure on $\mathcal{H}^{p,q}$, where each $\bar{\partial}_{tj}$ is defined by

$$(3.3) \quad \bar{\partial}_{tj} u : t \rightarrow \mathbb{H}^t \left(i_t^* [\bar{\partial}, \delta_{\bar{V}_j}] \mathbf{u} \right), \quad [\bar{\partial}, \delta_{\bar{V}_j}] := \bar{\partial} \delta_{\bar{V}_j} + \delta_{\bar{V}_j} \bar{\partial}.$$

Here \mathbf{u} is an arbitrary representative of $u \in \Gamma(\mathcal{H}^{p,q})$, \mathbb{H}^t denotes the orthogonal projection to $\mathcal{H}_t^{p,q}$ and V_j is an arbitrary smooth $(1, 0)$ -vector field on \mathcal{X} such that $\pi_* V_j = \partial/\partial t^j$.

Proof. First, let us show that $D^{0,1}$ is well defined. Since each $u^t \in \mathcal{H}_t^{p,q}$ is harmonic, we know that $i_t^* \bar{\partial} \mathbf{u} \equiv 0$, thus the definition of $\bar{\partial}_{tj} u$ does not depend on the choice of V_j . Thus it suffices to check that $\bar{\partial}_{tj} u$ does not depend on the choice of \mathbf{u} . Let \mathbf{u}' be another representative of u then we have

$$\mathbf{u} - \mathbf{u}' = \sum dt^j \wedge a_j + \sum d\bar{t}^k \wedge b_k.$$

Thus

$$(3.4) \quad i_t^* \delta_{\bar{V}_j} \bar{\partial}(\mathbf{u} - \mathbf{u}') = -i_t^* \bar{\partial} b_j,$$

which implies that

$$(3.5) \quad \mathbb{H}^t \left(i_t^* [\bar{\partial}, \delta_{\bar{V}_j}] (\mathbf{u} - \mathbf{u}') \right) = 0.$$

Thus $D^{0,1}$ is well defined. It is easy to check that

$$(3.6) \quad \bar{\partial}_{tj}(fu) = f \bar{\partial}_{tj} u + f_{\bar{j}} u,$$

where f is an arbitrary smooth function on B . By Newlander-Nirenberg's theorem, it suffices to show that $D^{0,1}$ is integrable, i.e. $(D^{0,1})^2 = 0$. By definition, it is sufficient to show

$$(3.7) \quad \bar{\partial}_{tj} \bar{\partial}_{tk} u = \bar{\partial}_{tk} \bar{\partial}_{tj} u,$$

for every $u \in \Gamma(\mathcal{H}^{p,q})$ and every $1 \leq j, k \leq m$.

Notice that

$$(3.8) \quad \mathbf{u}_j := [\bar{\partial}, \delta_{\bar{V}_j}] \mathbf{u},$$

satisfies (see Lemma 3.5, since $i_t^* \bar{\partial} \mathbf{u} \equiv 0$)

$$(3.9) \quad \bar{\partial}^t i_t^* \mathbf{u}_j = i_t^* \bar{\partial} \mathbf{u}_j = i_t^* [\bar{\partial}, \delta_{\bar{V}_j}] \bar{\partial} \mathbf{u} = 0.$$

Thus each $i_t^* \mathbf{u}_j$ has the following orthogonal decomposition

$$(3.10) \quad i_t^* \mathbf{u}_j = (\bar{\partial}_{tj} u)(t) + \bar{\partial}^t (\bar{\partial}^t)^* G^t(i_t^* \mathbf{u}_j),$$

where G^t is the Green operator on X_t . Using Kodaira-Spencer's theorem again, we know that $G^t(i_t^* \mathbf{u}_j)$ depends smoothly on t . Thus $(\bar{\partial}^t)^* G^t(i_t^* \mathbf{u}_j)$ depends smoothly on t . For each j , let us choose a smooth form \mathbf{v}_j on \mathcal{X} such that

$$i_t^* \mathbf{v}_j = (\bar{\partial}^t)^* G^t(i_t^* \mathbf{u}_j).$$

By definition, we know that each

$$(3.11) \quad \mathbf{u}_j - \bar{\partial} \mathbf{v}_j,$$

is a representative of $\bar{\partial}_{tj} u$. Thus we have

$$(3.12) \quad (\bar{\partial}_{tj} \bar{\partial}_{tk} - \bar{\partial}_{tk} \bar{\partial}_{tj}) u = \mathbb{H}^t \left(i_t^* [\bar{\partial}, \delta_{\bar{V}_j}] (\mathbf{u}_k - \bar{\partial} \mathbf{v}_k) - i_t^* [\bar{\partial}, \delta_{\bar{V}_k}] (\mathbf{u}_j - \bar{\partial} \mathbf{v}_j) \right)$$

$$(3.13) \quad = \mathbb{H}^t i_t^* \left([\bar{\partial}, \delta_{\bar{V}_j}] [\bar{\partial}, \delta_{\bar{V}_k}] - [\bar{\partial}, \delta_{\bar{V}_k}] [\bar{\partial}, \delta_{\bar{V}_j}] \right) \mathbf{u}.$$

Notice that

$$(3.14) \quad [L_{\bar{V}_j}, L_{\bar{V}_k}] = L_{[\bar{V}_j, \bar{V}_k]},$$

implies that

$$(3.15) \quad [\bar{\partial}, \delta_{\bar{V}_j}] [\bar{\partial}, \delta_{\bar{V}_k}] - [\bar{\partial}, \delta_{\bar{V}_k}] [\bar{\partial}, \delta_{\bar{V}_j}] = [\bar{\partial}, \delta_{[\bar{V}_j, \bar{V}_k]}]$$

Since each u^t is harmonic (thus $\bar{\partial}^t$ -closed), we have

$$(3.16) \quad (\bar{\partial}_{tj} \bar{\partial}_{tk} - \bar{\partial}_{tk} \bar{\partial}_{tj}) u = \mathbb{H}^t i_t^* [\bar{\partial}, \delta_{[\bar{V}_j, \bar{V}_k]}] \mathbf{u} = \mathbb{H}^t i_t^* \delta_{[\bar{V}_j, \bar{V}_k]} \bar{\partial} \mathbf{u} = \mathbb{H}^t \delta_{[\bar{V}_j, \bar{V}_k]|_{X_t}} \bar{\partial}^t u^t = 0.$$

The proof is complete. \square

3.2. Chern connection on $\mathcal{H}^{p,q}$. In this subsection, we shall define the Chern connection on $\mathcal{H}^{p,q}$.

Assume that \mathcal{X} possesses a Kähler form, say ω . Then each fibre X_t possesses a Kähler form $\omega^t := \omega|_{X_t}$. Recall that a smooth k -form α on X_t is said to be primitive with respect to ω^t if $k \leq n$ and $\omega_{n-k+1}^t \wedge \alpha = 0$ on X_t . Let $u : t \mapsto u^t \in \mathcal{H}_t^{p,q}$ be a smooth section of $\mathcal{H}^{p,q}$. By the Lefschetz decomposition theorem, each u^t has a unique decomposition of the form

$$(3.17) \quad u^t := \sum_r \omega_r^t \wedge u_r^t,$$

where each u_r^t is a smooth E_t -valued primitive $(p-r, q-r)$ -form. Let us denote by $*$ the Hodge-Poincaré-de Rham star operator with respect to ω^t . Then we have

$$(3.18) \quad * u^t := \sum_r C_r \omega_{n+r-p-q}^t \wedge u_r^t, \quad C_r = i^{(p+q-2r)^2} (-1)^{p-r}.$$

Since u^t depends smoothly on t , we know that each u_r^t also depends smoothly on t . Thus for each r , there exists a smooth E -valued $(p-r, q-r)$ -form, say \mathbf{u}_r on \mathcal{X} such that

$$(3.19) \quad \mathbf{u}_r|_{X_t} = u_r^t.$$

By Definition 3.1, we know that

$$(3.20) \quad \sum_r \omega_r \wedge \mathbf{u}_r$$

is a representative of u . We shall use the following definition:

Definition 3.2. We call a smooth E -valued $(n-q, n-p)$ -form \mathbf{u}^* on \mathcal{X} a dual-representative of $u \in \Gamma(\mathcal{H}^{p,q})$ if

$$(3.21) \quad \mathbf{u}^* = \sum_r C_r \omega_{n+r-p-q} \wedge \mathbf{u}_r.$$

By definition, we know that if \mathbf{u} is a representative of $u \in \Gamma(\mathcal{H}^{p,q})$ and \mathbf{v}^* is a dual-representative of $v \in \Gamma(\mathcal{H}^{p,q})$ then

$$(3.22) \quad (u, v) = \pi_* \{ \mathbf{u}, \mathbf{v}^* \},$$

where $\{ \cdot, \cdot \}$ is the canonical sesquilinear pairing (see page 268 in [13]). Now we have (see page 12 in [30])

$$(3.23) \quad \frac{\partial}{\partial t^j} (u, v) = \frac{\partial}{\partial t^j} \pi_* \{ \mathbf{u}, \mathbf{v}^* \}$$

$$(3.24) \quad = \pi_* (L_{V_j} \{ \mathbf{u}, \mathbf{v}^* \})$$

$$(3.25) \quad = \pi_* (\{ L_j \mathbf{u}, \mathbf{v}^* \} + \{ \mathbf{u}, L_{\bar{j}} \mathbf{v}^* \}),$$

where V_j is an arbitrary smooth $(1,0)$ -vector field on \mathcal{X} such that $\pi_* V_j = \partial / \partial t^j$ and

$$(3.26) \quad L_j := d^E \delta_{V_j} + \delta_{V_j} d^E, \quad L_{\bar{j}} := d^E \delta_{\bar{V}_j} + \delta_{\bar{V}_j} d^E.$$

Here $d^E = \partial^E + \bar{\partial}$ denotes the Chern connection on E . In order to find a good expression of the Chern connection on $\mathcal{H}^{p,q}$, we shall introduce the following definition:

Definition 3.3. Assume that \mathcal{X} possesses a Kähler form ω . A smooth $(1,0)$ -vector field V on \mathcal{X} is said to be horizontal with respect to ω if

$$(3.27) \quad i_t^* (\delta_{\bar{V}} \omega) = 0,$$

on X_t for every $t \in B$. Moreover, for each $1 \leq j \leq m$, we call V_j the horizontal lift vector field of $\partial / \partial t^j$ with respect to ω if V_j is horizontal with respect to ω and satisfies

$$(3.28) \quad \pi_* (V_j) = \partial / \partial t^j.$$

Now we can prove that:

Proposition 3.2. Assume that \mathcal{X} possesses a Kähler form ω and each V_j is the horizontal lift vector field of $\partial / \partial t^j$. Then

$$(3.29) \quad \pi_* \{ \mathbf{u}, L_{\bar{j}} \mathbf{v}^* \} = (u, \bar{\partial}_{t^j} v),$$

for every smooth sections u, v of $\mathcal{H}^{p,q}$.

Proof. For bidegree reason, we have

$$(3.30) \quad \pi_* \{ \mathbf{u}, L_{\bar{j}} \mathbf{v}^* \} = \pi_* \{ \mathbf{u}, [\bar{\partial}, \delta_{\bar{V}_j}] \mathbf{v}^* \},$$

Thus it suffices to show that

$$(3.31) \quad (\bar{\partial}_{t^j} v)(t) = \mathbb{H}^t \left((-1)^{p+q} * i_t^* [\bar{\partial}, \delta_{\bar{V}_j}] \mathbf{v}^* \right).$$

By Theorem 3.1 and Definition 3.2, it suffices to check that

$$(3.32) \quad i_t^*[[\bar{\partial}, \delta_{V_j}], \omega] = 0.$$

Since each V_j is horizontal, the above equality is always true. The proof is complete. \square

By Proposition 3.2, we have

$$(3.33) \quad \frac{\partial}{\partial t^j}(u, v) = \pi_*\{L_j \mathbf{u}, \mathbf{v}^*\} + (u, \bar{\partial}_{t^j} v).$$

By definition, the $(1, 0)$ -part of the Chern connection $D^{1,0} = \sum dt^j \otimes D_{t^j}$ should satisfy

$$(3.34) \quad \frac{\partial}{\partial t^j}(u, v) = (D_{t^j} u, v) + (u, \bar{\partial}_{t^j} v).$$

Thus we have:

Proposition 3.3. *Assume that \mathcal{X} possesses a Kähler form ω and each V_j is the horizontal lift vector field of $\partial/\partial t^j$. Then the $(1, 0)$ -part of the Chern connection on $\mathcal{H}^{p,q}$ satisfies*

$$(3.35) \quad D_{t^j} u : t \mapsto \mathbb{H}^t(i_t^*[\partial^E, \delta_{V_j}]\mathbf{u}),$$

where \mathbf{u} is an arbitrary representative of $u \in \Gamma(\mathcal{H}^{p,q})$.

3.3. Curvature of $\mathcal{H}^{p,q}$. In this section, we shall assume that \mathcal{X} possesses a Kähler form, say ω , and $\mathcal{H}^{p,q}$ satisfies **A**. For each $1 \leq j \leq m$, we shall denote by V_j the horizontal lift vector field of $\partial/\partial t^j$ with respect to ω .

Let u, v be two holomorphic sections of $\mathcal{H}^{p,q}$. By Proposition 3.2 and (3.25), we have

$$(3.36) \quad \frac{\partial}{\partial t^j}(u, v) = \pi_*\{L_j \mathbf{u}, \mathbf{v}^*\}.$$

Thus we have

$$(3.37) \quad (u, v)_{j\bar{k}} = \frac{\partial}{\partial t^k} \pi_*\{L_j \mathbf{u}, \mathbf{v}^*\} = \pi_* L_{\bar{V}_k} \{L_j \mathbf{u}, \mathbf{v}^*\}$$

$$(3.38) \quad = \pi_* (\{L_{\bar{k}} L_j \mathbf{u}, \mathbf{v}^*\} + \{L_j \mathbf{u}, L_k \mathbf{v}^*\})$$

$$(3.39) \quad = \pi_* (\{[L_{\bar{k}}, L_j] \mathbf{u}, \mathbf{v}^*\} + \{L_j L_{\bar{k}} \mathbf{u}, \mathbf{v}^*\} + \{L_j \mathbf{u}, L_k \mathbf{v}^*\}).$$

Since u is a holomorphic section, by Theorem 3.1, for bidegree reason, we have

$$(3.40) \quad \pi_*\{L_{\bar{k}} \mathbf{u}, \mathbf{v}^*\} = 0.$$

Thus

$$(3.41) \quad 0 = \frac{\partial}{\partial t^j} \pi_*\{L_{\bar{k}} \mathbf{u}, \mathbf{v}^*\} = \pi_* (\{L_j L_{\bar{k}} \mathbf{u}, \mathbf{v}^*\} + \{L_{\bar{k}} \mathbf{u}, L_{\bar{j}} \mathbf{v}^*\}).$$

By (3.39), we have

$$(3.42) \quad (u, v)_{j\bar{k}} = \pi_* (\{[L_{\bar{k}}, L_j] \mathbf{u}, \mathbf{v}^*\} - \{L_{\bar{k}} \mathbf{u}, L_{\bar{j}} \mathbf{v}^*\} + \{L_j \mathbf{u}, L_k \mathbf{v}^*\}).$$

For the last term, since each V_j is horizontal, by (3.18), we have

$$\begin{aligned} \pi_*\{L_j \mathbf{u}, L_k \mathbf{v}^*\} &= \pi_* (\{[\partial^E, \delta_{V_j}] \mathbf{u}, [\partial^E, \delta_{V_k}] \mathbf{v}^*\} + \{[\bar{\partial}, \delta_{V_j}] \mathbf{u}, [\bar{\partial}, \delta_{V_k}] \mathbf{v}^*\}) \\ &= (i_t^*[\partial^E, \delta_{V_j}] \mathbf{u}, i_t^*[\partial^E, \delta_{V_k}] \mathbf{v}) - (\bar{\partial} V_j|_{X_t \lrcorner} u^t, \bar{\partial} V_k|_{X_t \lrcorner} v^t). \end{aligned}$$

By the same reason, we have

$$(3.43) \quad \pi_*\{L_{\bar{k}} \mathbf{u}, L_{\bar{j}} \mathbf{v}^*\} = (i_t^*[\bar{\partial}, \delta_{\bar{V}_k}] \mathbf{u}, i_t^*[\bar{\partial}, \delta_{\bar{V}_j}] \mathbf{v}) - (\partial \bar{V}_k|_{X_t \lrcorner} u^t, \partial \bar{V}_j|_{X_t \lrcorner} v^t)$$

By definition of the Chern connection, we have

$$(3.44) \quad (\Theta_{j\bar{k}}u, v) = (D_{t^j}u, D_{t^k}v) - (u, v)_{j\bar{k}}.$$

Put

$$(3.45) \quad a_j^u = D_{t^j}u - i_t^*[\partial^E, \delta_{V_j}]\mathbf{u}; \quad b_j^u = \bar{\partial}V_j|_{X_t \lrcorner} u^t,$$

and

$$(3.46) \quad a_j^v := i_t^*[\bar{\partial}, \delta_{\bar{V}_j}]\mathbf{v}, \quad b_j^v := \partial\bar{V}_j|_{X_t \lrcorner} v^t.$$

Then we have:

Theorem 3.4. *Assume that \mathcal{X} possesses a Kähler form ω and $\mathcal{H}^{p,q}$ satisfies **A**. Let u and v be holomorphic sections of $\mathcal{H}^{p,q}$. Then we have*

$$(3.47) \quad (\Theta_{j\bar{k}}u, v) = (b_j^u, b_k^v) - (a_j^u, a_k^v) + \pi_*\{[L_j, L_{\bar{k}}]\mathbf{u}, \mathbf{v}^*\} + (a_k^u, a_j^v) - (b_k^u, b_j^v).$$

Remark: For the middle term in the above formula, we shall use

$$(3.48) \quad [L_j, L_{\bar{k}}] = [d^E, \delta_{[V_j, \bar{V}_k]}] + \Theta(E, h)(V_j, \bar{V}_k).$$

In order to study the other terms, we have to use Hörmander's L^2 -theory [15] for the generalized $\bar{\partial}$ -equation.

3.4. Generalized $\bar{\partial}$ -equation associated to the curvature formula. We shall use the following lemma:

Lemma 3.5. *If both \mathbf{u} and \mathbf{u}' are representatives of u then*

$$(3.49) \quad i_t^*L_{\bar{k}}(\mathbf{u} - \mathbf{u}') = 0, \quad i_t^*L_j(\mathbf{u} - \mathbf{u}') = 0.$$

Proof. By definition, we have

$$(3.50) \quad \mathbf{u} - \mathbf{u}' = \sum dt^j \wedge a_j + \sum d\bar{t}^k \wedge b_k.$$

Since

$$(3.51) \quad (d^E \delta_V + \delta_V d^E)(df \wedge a) = d(Vf) \wedge a + df \wedge (d^E \delta_V + \delta_V d^E)a.$$

Apply this formula to $f = t^j, \bar{t}^k$, $V = V_j, \bar{V}_k$ and $a = a_j, b_k$, we get (3.49). \square

Now we can prove:

Lemma 3.6. *With the notation in Theorem 3.4, we have*

$$(3.52) \quad (\bar{\partial}^t)^* a_j^u = (\bar{\partial}^t)^* b_j^v = 0, \quad \forall t \in B, \quad \forall 1 \leq j \leq m.$$

Moreover, if $\mathcal{H}_t^{p,q} \subset \ker \partial^{E_t}$ for every $t \in B$ then

$$(3.53) \quad \bar{\partial}^t a_j^u = \partial^{E_t} b_j^u + c_j^u, \quad \partial^{E_t} a_j^v = -\bar{\partial}^t b_j^v - c_j^v,$$

and each a_j^u is the L^2 -minimal solution of (3.53), where

$$(3.54) \quad c_j^u := (V_j \lrcorner \Theta(E, h))|_{X_t} \wedge u^t, \quad c_j^v := (\bar{V}_j \lrcorner \Theta(E, h))|_{X_t} \wedge v^t.$$

Assume further that $\mathcal{H}_t^{p,q} = \ker \partial^{E_t} \cap \ker (\partial^{E_t})^*$ for every $t \in B$ then each a_j^v is also the L^2 -minimal solution of (3.53).

Proof. By (3.45), we have

$$(3.55) \quad (\bar{\partial}^t)^* a_j^u = -(\bar{\partial}^t)^* (i_t^* [\partial^E, \delta_{V_j}] \mathbf{u}).$$

Since $(\bar{\partial}^t)^* = - * \partial^{E_t} *$, by (3.18), the following equality

$$(3.56) \quad i_t^* (\partial^E [\partial^E, \delta_{V_j}] \mathbf{u}^*) = 0.$$

implies $(\bar{\partial}^t)^* a = 0$. Notice that

$$(3.57) \quad \partial^E [\partial^E, \delta_{V_j}] = [\partial^E, \delta_{V_j}] \partial^E,$$

and

$$(3.58) \quad i_t^* \partial^E \mathbf{u}^* = \partial^{E_t} * u^t = 0.$$

Thus (3.56) follows from (3.51). By the same proof, we have $(\bar{\partial}^t)^* b_j^v = 0$, thus (3.52) is true. Now let us prove (3.53). By (3.45), we have

$$(3.59) \quad \bar{\partial}^t a_j^u - \partial^{E_t} b_j^u = -i_t^* (\bar{\partial} [\partial^E, \delta_{V_j}] \mathbf{u} + \partial^E [\bar{\partial}, \delta_{V_j}] \mathbf{u}).$$

Since by our assumption, $\mathcal{H}_t^{p,q} \subset \ker \partial^{E_t}$, thus we have

$$(3.60) \quad i_t^* (\partial^E \mathbf{u}) = 0, \quad i_t^* (\bar{\partial} \mathbf{u}) = 0,$$

by (3.51), we have

$$(3.61) \quad i_t^* [\partial^E, \delta_{V_j}] \bar{\partial} \mathbf{u} = i_t^* [\bar{\partial}, \delta_{V_j}] \partial^E \mathbf{u} = 0.$$

Thus

$$(3.62) \quad \bar{\partial}^t a_j^u - \partial^{E_t} b_j^u = -i_t^* ([\bar{\partial}, [\partial^E, \delta_{V_j}]] \mathbf{u} + [\partial^E, [\bar{\partial}, \delta_{V_j}]] \mathbf{u}) = i_t^* [\delta_{V_j}, [\bar{\partial}, \partial^E]] \mathbf{u} = c_j^u.$$

By the same proof, we have

$$\partial^{E_t} a_j^v = -\bar{\partial}^t b_j^v - c_j^v.$$

Thus (3.53) is true. Now let us prove the last part. Since v is a holomorphic section, by Theorem 3.1, we have that each a_j^v has no $\bar{\partial}^t$ -harmonic part. By our assumption, each $\bar{\partial}^t$ -harmonic space is equal to the ∂^{E_t} -harmonic space, we know that each a_j^v has no ∂^{E_t} -harmonic part. Thus it is sufficient to prove that each a_j^v is $(\partial^{E_t})^*$ -closed. Since $(\partial^{E_t})^* = - * \bar{\partial}^t *$, it is sufficient to show that

$$(3.63) \quad \bar{\partial}^t * a_j^v = 0.$$

By (3.18), we know that

$$(3.64) \quad \bar{\partial}^t * a_j^v = \bar{\partial}^t i_t^* [\bar{\partial}, \delta_{V_j}] \mathbf{v}^* = i_t^* [\bar{\partial}, \delta_{V_j}] \bar{\partial} \mathbf{v}^*.$$

Moreover, by our assumption, each v^t is also in the ∂^{E_t} -harmonic space. Thus we have

$$(3.65) \quad i_t^* \bar{\partial} \mathbf{v}^* = \bar{\partial}^t * v^t = 0,$$

by (3.51) and (3.64), we know that $\bar{\partial}^t * a_j^v = 0$. The proof is complete. \square

By (3.47) and the above lemma, the L^2 -estimates for the generalized $\bar{\partial}$ -equation (see (3.53)) determine the positivity of $\mathcal{H}^{p,q}$. We shall show how to use the following version of Hörmander's L^2 -theory [15] to study the curvature of $\mathcal{H}^{p,q}$:

Theorem 3.7. *Let (E, h) be a Hermitian vector bundle over an n -dimensional compact complex manifold X . Let $q \geq 0$ be an integer. Assume that X possesses a Hermitian metric ω such that $\omega_{\max\{q,1\}}$ is $\bar{\partial}$ -closed. Let v be a smooth $\bar{\partial}$ -closed E -valued $(n, q+1)$ -form. Assume that*

$$i\Theta(E, h) \wedge \omega_q > 0 \text{ on } X, \text{ (resp. } i\Theta(E, h) \wedge \omega_q \equiv 0 \text{ on } X),$$

and

$$I(v) := \inf_{v=\partial^E b+c} \|b\|_\omega^2 + ([i\Theta(E, h), \Lambda_\omega]^{-1}c, c)_\omega < \infty, \text{ (resp. } I(v) := \inf_{v=\partial^E b} \|b\|_\omega^2 < \infty).$$

Then there exists a smooth E -valued (n, q) -form a such that $\bar{\partial}a = v$ and

$$(3.66) \quad \|a\|_\omega^2 \leq I(v).$$

Proof. Let γ be an arbitrary E -valued smooth $(n-q-1, 0)$ -form on X . Put

$$(3.67) \quad T = i^{(n-q-1)^2} \{\gamma, \gamma\} \wedge \omega_q, \quad u = \gamma \wedge \omega_{q+1}.$$

Since $\bar{\partial}\omega_q = 0$, one may check that $i\partial\bar{\partial}T$ can be written as

$$(3.68) \quad -2\operatorname{Re}\langle \bar{\partial}\bar{\partial}^*u, u \rangle \omega_n + |\bar{\partial}^*u|^2 \omega_n + i^{(n-q-1)^2} \{i\Theta(E, h)\gamma, \gamma\} \wedge \omega_q - S,$$

where

$$(3.69) \quad S = i^{(n-q)^2} \{\bar{\partial}\gamma, \bar{\partial}\gamma\} \wedge \omega_q.$$

By Lemma 4.2 in Berndtsson's lecture notes [6], we have

$$(3.70) \quad S = (|\omega_{q+1} \wedge \bar{\partial}\gamma|^2 - |\bar{\partial}\gamma|^2) \omega_n.$$

Since $\omega_{\max\{q,1\}}$ is $\bar{\partial}$ -closed, we have $\bar{\partial}\omega_{q+1} = 0$, thus

$$(3.71) \quad S = (|\bar{\partial}u|^2 - |(\partial^E)^*u|^2) \omega_n.$$

Notice that $\int_X i\partial\bar{\partial}T = 0$, thus we have

$$(3.72) \quad \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 - \|(\partial^E)^*u\|^2 = \int_X i^{(n-q-1)^2} \{i\Theta(E, h)\gamma, \gamma\} \wedge \omega_q$$

$$(3.73) \quad = ([i\Theta(E, h), \Lambda_\omega]u, u),$$

where Λ_ω is the adjoint of $\omega \wedge \cdot$. By Hörmander's L^2 theory, it suffices to show that

$$(3.74) \quad |(v, u)|^2 \leq I(v)(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2), \quad \forall \text{ smooth } u,$$

which follows from (3.72) and (3.73) by the Cauchy-Schwartz inequality. \square

We shall also use the following generalized version of Theorem 3.7 (see Remark 13.5 in [11] for related results):

Theorem 3.8. *Let (E, h) be a Hermitian vector bundle over an n -dimensional compact complex manifold X . Let $q \geq 0$ be an integer. Assume that X possesses a Hermitian metric ω such that $\omega_{\max\{q,1\}}$ is $\bar{\partial}$ -closed. Let v be a smooth $\bar{\partial}$ -closed E -valued $(n, q+1)$ -form. Assume that $i\Theta(E, h) \wedge \omega_q \geq 0$ on X and and*

$$I(v) := \inf_{v=\partial^E b+c} \left\{ \|b\|_\omega^2 + \lim_{\varepsilon \rightarrow 0} (([i\Theta(E, h), \Lambda_\omega] + \varepsilon)^{-1}c, c)_\omega \right\} < \infty.$$

Then there exists a smooth E -valued (n, q) -form a such that $\bar{\partial}a = v$ and

$$(3.75) \quad \|a\|_\omega^2 \leq I(v).$$

Proof. By (3.72) and (3.73), for every $\varepsilon > 0$, we have

$$(3.76) \quad |(v, u)|^2 \leq I(v)(\|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2 + \varepsilon\|u\|^2), \quad \forall \text{ smooth } u,$$

By Hörmander's L^2 theory, there exist a_ε and α_ε such that

$$(3.77) \quad \bar{\partial}a_\varepsilon + \sqrt{\varepsilon}\alpha_\varepsilon = v,$$

and

$$(3.78) \quad \|a_\varepsilon\|^2 + \|\alpha_\varepsilon\|^2 \leq I(v).$$

Thus $\sqrt{\varepsilon}\alpha_\varepsilon \rightarrow 0$ in the sense of current as $\varepsilon \rightarrow 0$. By taking a weak limit, say a' , of a_ε , we know that there exists a' such that $\bar{\partial}a' = v$ in the sense of current and

$$(3.79) \quad \|a'\|^2 \leq I(v).$$

Let a be the L^2 -minimal solution. Then a fits our needs. \square

3.5. Curvature of $\mathcal{H}^{n,q}$. Theorem 3.8 implies that there is a good L^2 -estimate for (3.53) if $p = n$. We shall show how to use it to prove Theorem 1.1 in this subsection.

Let us study the middle term in (3.47) first. By (3.48), we have

$$(3.80) \quad \pi_*\{[L_j, L_{\bar{k}}]\mathbf{u}_j, \mathbf{u}_k^*\} = (\Theta(E, h)(V_j, \bar{V}_k)u_j, u_k) + (\partial^{E_t}\delta_{[V_j, \bar{V}_k]}u_j, u_k).$$

The first term is clear. For the second term, we need the following lemma:

Lemma 3.9. $\sum(\partial^{E_t}\delta_{[V_j, \bar{V}_k]}u_j, u_k)$ can be written as $A + Nak$, where

$$(3.81) \quad A := \sum i^{(n-q)^2} \pi_*\{c_{j\bar{k}}(\omega)\omega_{q-1} \wedge i\Theta(E, h)\mathbf{u}_j^*, \mathbf{u}_k^*\}, \quad c_{j\bar{k}}(\omega) := \langle V_j, V_k \rangle_\omega,$$

$$\text{and } Nak := \sum(c_{j\bar{k}}(\omega)\bar{\partial}^t * u_j^t, \bar{\partial}^t * u_k^t) \geq 0.$$

We shall prove the above Lemma later. By the above lemma, we have

$$(3.82) \quad \sum \pi_*\{[L_j, L_{\bar{k}}]\mathbf{u}_j, \mathbf{u}_k^*\} = A + B + Nak,$$

where

$$(3.83) \quad B := \sum (\Theta(E, h)(V_j, \bar{V}_k)u_j, u_k).$$

Thus by (3.47), put

$$(3.84) \quad b = \sum b_j^{u_j}, \quad a = \sum a_j^{u_j},$$

then we have (notice that $b_j^{u_k} = 0$ since $p = n$)

$$(3.85) \quad \sum(\Theta_{j\bar{k}}u_j, u_k) = \|b\|^2 - \|a\|^2 + A + B + Nak + Gri,$$

where

$$(3.86) \quad Gri := \sum(a_k^{u_j}, a_j^{u_k}),$$

is non-negative if $u_j = \xi_j u$. Put

$$(3.87) \quad C_\varepsilon := ([i\Theta(E^t, h), \Lambda_{\omega^t}] + \varepsilon)^{-1}c, c) , \quad c := \sum c_j^{u_j}.$$

Then we have

$$(3.88) \quad \sum(\Theta_{j\bar{k}}u_j, u_k) = A + B - C_\varepsilon + Nak + Gri + R_\varepsilon,$$

where

$$(3.89) \quad R_\varepsilon := ([i\Theta(E^t, h), \Lambda_{\omega^t}] + \varepsilon)^{-1}c, c) + \|b\|^2 - \|a\|^2.$$

Proof of Theorem 1.1 By (3.53) and Theorem 3.8, we know that R_ε is always non-negative. In order to show that $\mathcal{H}^{n,q}$ is Griffiths semi-positive, it suffices to prove

$$\lim_{\varepsilon \rightarrow 0} (A + B - C_\varepsilon) \geq 0,$$

which follows from the following lemma:

Lemma 3.10. *If $\Theta(E, h)$ is q -semipositive with respect to ω then*

$$(3.90) \quad A + B + \varepsilon \sum (c_{j\bar{k}}(\omega) u_j, u_k) - C_\varepsilon \geq 0, \quad \forall \varepsilon > 0.$$

Proof. Put

$$(3.91) \quad I_\varepsilon := A + B + \varepsilon \sum (c_{j\bar{k}}(\omega) u_j, u_k) - C_\varepsilon,$$

and

$$(3.92) \quad T_\varepsilon := \omega_q \wedge i\Theta(E, h) + \varepsilon \omega_{q+1} \otimes Id_E, \quad T_\varepsilon^t := T_\varepsilon|_{X_t}.$$

We claim that for each j one may choose a dual representative \mathbf{u}_j^* of u_j such that

$$(3.93) \quad i_t^* \delta_{V_j} (T_\varepsilon \wedge \mathbf{u}_j^*) \equiv 0,$$

on X_t for every $t \in B$ and

$$(3.94) \quad I_\varepsilon \wedge i^{m^2} dt \wedge \bar{dt} = \pi_* (c_q \{T_\varepsilon \wedge \mathbf{u}^*, \mathbf{u}^*\}) \geq 0, \quad \mathbf{u}^* := \sum \mathbf{u}_j^* \wedge \delta_{V_j} dt,$$

where $c_q := i^{(m+n-q-1)^2}$ and $dt := dt^1 \wedge \dots \wedge dt^m$.

In fact, (3.93) is equivalent to

$$(3.95) \quad \omega_q^t \wedge (V_j \lrcorner i\Theta(E, h))|_{X_t} \wedge *u_j^t + T_\varepsilon^t \wedge i_t^* \delta_{V_j} \mathbf{u}_j^* \equiv 0.$$

Notice that

$$(3.96) \quad Q_\varepsilon^{-1} (T_\varepsilon^t \wedge i_t^* \delta_{V_j} \mathbf{u}_j^*) = \omega_{q+1}^t \wedge i_t^* \delta_{V_j} \mathbf{u}_j^*, \quad Q_\varepsilon := [\Theta(E^t, h), \Lambda_{\omega^t}] + \varepsilon,$$

Thus there exists \mathbf{u}_j^* such that (3.93) is true. Now choose \mathbf{u}_j^* that satisfies (3.93), then

$$(3.97) \quad c = \sum i^{(n-q)^2+1} T_\varepsilon^t \wedge i_t^* \delta_{V_j} \mathbf{u}_j^*,$$

which implies

$$(3.98) \quad Q_\varepsilon^{-1} c = \sum i^{(n-q)^2+1} \omega_{q+1}^t \wedge i_t^* \delta_{V_j} \mathbf{u}_j^*.$$

Thus we have

$$(3.99) \quad C_\varepsilon = (Q_\varepsilon^{-1} c, c) = \pi_* \left(i^{(n-q-1)^2} \{T_\varepsilon \wedge \sum \delta_{V_j} \mathbf{u}_j^*, \sum \delta_{V_j} \mathbf{u}_j^*\} \right).$$

Recall that

$$(3.100) \quad A = \sum i^{(n-q)^2} \pi_* \{c_{j\bar{k}}(\omega) \omega_{q-1} \wedge i\Theta(E, h) \mathbf{u}_j^*, \mathbf{u}_k^*\},$$

and

$$(3.101) \quad B = \sum i^{(n-q)^2} \pi_* \{\omega_q \wedge \Theta(E, h)(V_j, \bar{V}_k) \mathbf{u}_j^*, \mathbf{u}_k^*\}.$$

Notice that

$$(3.102) \quad i_t^* \left(i \delta_{V_j} \delta_{\bar{V}_k} T_\varepsilon \right) = \omega_q^t \wedge \Theta(E, h)(V_j, \bar{V}_k) + i\Theta(E, h) \wedge c_{j\bar{k}}(\omega) \omega_{q-1}^t + \varepsilon c_{j\bar{k}}(\omega) \omega_q^t.$$

Thus

$$(3.103) \quad A + B + \varepsilon \sum (c_{j\bar{k}}(\omega) u_j, u_k) = \sum i^{(n-q)^2} \pi_* \left\{ (i \delta_{V_j} \delta_{\bar{V}_k} T_\varepsilon) \wedge \mathbf{u}_j^*, \mathbf{u}_k^* \right\}.$$

Notice that

$$\begin{aligned} \delta_{V_j} \delta_{\bar{V}_k} \{T_\varepsilon \wedge \mathbf{u}_j^*, \mathbf{u}_k^*\} &= (-1)^{n-q} \{(\delta_{V_j} T_\varepsilon) \wedge \mathbf{u}_j^*, \delta_{V_k} \mathbf{u}_k^*\} - \{(\delta_{\bar{V}_k} T_\varepsilon) \wedge \delta_{V_j} \mathbf{u}_j^*, \mathbf{u}_k^*\} \\ &\quad + \{(\delta_{V_j} \delta_{\bar{V}_k} T_\varepsilon) \wedge \mathbf{u}_j^*, \mathbf{u}_k^*\} + (-1)^{n-q} \{T_\varepsilon \wedge \delta_{V_j} \mathbf{u}_j^*, \delta_{V_k} \mathbf{u}_k^*\}. \end{aligned}$$

By (3.93), we have

$$(3.104) \quad \pi_* \{(\delta_{V_j} T_\varepsilon) \wedge \mathbf{u}_j^*, \delta_{V_k} \mathbf{u}_k^*\} = -\pi_* \{T_\varepsilon \wedge \delta_{V_j} \mathbf{u}_j^*, \delta_{V_k} \mathbf{u}_k^*\},$$

and

$$(3.105) \quad \pi_* \{(\delta_{\bar{V}_k} T_\varepsilon) \wedge \delta_{V_j} \mathbf{u}_j^*, \mathbf{u}_k^*\} = (-1)^{n-q} \pi_* \{T_\varepsilon \wedge \delta_{V_j} \mathbf{u}_j^*, \delta_{V_k} \mathbf{u}_k^*\}.$$

Thus

$$(3.106) \quad \pi_* \delta_{V_j} \delta_{\bar{V}_k} \{T_\varepsilon \wedge \mathbf{u}_j^*, \mathbf{u}_k^*\} = \pi_* \{(\delta_{V_j} \delta_{\bar{V}_k} T_\varepsilon) \wedge \mathbf{u}_j^*, \mathbf{u}_k^*\} - (-1)^{n-q} \pi_* \{T_\varepsilon \wedge \delta_{V_j} \mathbf{u}_j^*, \delta_{V_k} \mathbf{u}_k^*\},$$

which implies that

$$(3.107) \quad \sum i^{(n-q)^2} \pi_* \left(i \delta_{V_j} \delta_{\bar{V}_k} \{T_\varepsilon \wedge \mathbf{u}_j^*, \mathbf{u}_k^*\} \right) = I_\varepsilon.$$

Notice that for bi-degree reason, we have

$$(3.108) \quad (\delta_{V_j} \delta_{\bar{V}_k} \{T_\varepsilon \wedge \mathbf{u}_j^*, \mathbf{u}_k^*\}) \wedge dt \wedge \overline{dt} = (-1)^m \{T_\varepsilon \wedge \mathbf{u}_j^*, \mathbf{u}_k^*\} \wedge \delta_{V_j} dt \wedge \overline{\delta_{V_k} dt}.$$

Thus (3.94) is true. The proof is complete. \square

Proof of the remark behind Theorem 1.1: Since $p = n$, we know that

$$(3.109) \quad b_j^{u_k} = c_j^{u_k} = 0.$$

Thus by Lemma 3.6, if $\mathcal{H}_t^{p,q} = \ker \partial^{E_t} \cap \ker (\partial^{E_t})^*$ for every $t \in B$ then we have

$$(3.110) \quad a_j^{u_k} = 0,$$

which implies that $Gri = 0$. Thus $\mathcal{H}^{n,q}$ is Nakano semi-positive.

Proof of Lemma 3.9: Put

$$(3.111) \quad I_{jk} := (\partial^{E_t} \delta_{[V_j, \bar{V}_k]} u_j, u_k).$$

Denote by V the $(1,0)$ -part of $[V_j, \bar{V}_k]$, then we have

$$(3.112) \quad I_{jk} = \int_{X_t} \{\partial^{E_t} \delta_V u_j^t, * u_k^t\} = (-1)^{n-q} \int_{X_t} \{\delta_V u_j^t, \bar{\partial}^t * u_k^t\}.$$

Since

$$(3.113) \quad u_j^t = i^{(n-q)^2} \omega_q^t \wedge * u_j^t,$$

and

$$(3.114) \quad i^{(n-q)^2} \omega_q^t \wedge \bar{\partial}^t * u_k^t = \bar{\partial}^t u_k^t = 0,$$

we have

$$(3.115) \quad I_{jk} = (-1)^{n-q} i^{(n-q)^2} \int_{X_t} \{(\delta_V \omega_q^t) \wedge * u_j^t, \bar{\partial}^t * u_k^t\}.$$

By definition, $\delta_V \omega_q^t$ is the $(q-1, q)$ -part of

$$(3.116) \quad i_t^* ((L_{V_j} \bar{V}_k) \lrcorner \omega_q).$$

Since

$$(3.117) \quad (L_{V_j} \bar{V}_k) \lrcorner \omega_q = L_{V_j} (\bar{V}_k \lrcorner \omega_q) - \bar{V}_k \lrcorner L_{V_j} \omega_q,$$

and

$$(3.118) \quad \bar{V}_k \lrcorner \omega_q = -i \sum c_{j\bar{k}}(\omega) dt^j \wedge \omega_{q-1}, \quad L_{V_j} \omega_q = d \left(i \sum c_{j\bar{k}}(\omega) dt^{\bar{k}} \wedge \omega_{q-1} \right).$$

Thus

$$(3.119) \quad i_t^* ((L_{V_j} \bar{V}_k) \lrcorner \omega_q) = i_t^* d (i c_{j\bar{k}}(\omega) \wedge \omega_{q-1}),$$

and

$$(3.120) \quad \delta_V \omega_q^t = \bar{\partial}^t (i c_{j\bar{k}}(\omega) \wedge \omega_{q-1}^t).$$

Now I_{jk} can be written as

$$i^{(n-q+1)^2} \int_{X_t} \{ \bar{\partial}^t (c_{j\bar{k}}(\omega) \wedge \omega_{q-1}^t \wedge *u_j^t), \bar{\partial}^t * u_k^t \} - \{ c_{j\bar{k}}(\omega) \wedge \omega_{q-1}^t \wedge \bar{\partial}^t * u_j^t, \bar{\partial}^t * u_k^t \}.$$

By (3.114), each $\bar{\partial}^t * u_k^t$ is primitive, thus

$$(3.121) \quad - \sum i^{(n-q+1)^2} \int_{X_t} \{ c_{j\bar{k}}(\omega) \wedge \omega_{q-1}^t \wedge \bar{\partial}^t * u_j^t, \bar{\partial}^t * u_k^t \} = Nak.$$

Now it suffices to show

$$(3.122) \quad A = \sum i^{(n-q+1)^2} \int_{X_t} \{ \bar{\partial}^t (c_{j\bar{k}}(\omega) \wedge \omega_{q-1}^t \wedge *u_j^t), \bar{\partial}^t * u_k^t \}.$$

Notice that the right hand side can be written as

$$(3.123) \quad \sum i^{(n-q)^2} (-i) \int_{X_t} \{ c_{j\bar{k}}(\omega) \wedge \omega_{q-1}^t \wedge *u_j^t, \partial^{E_t} \bar{\partial}^t * u_k^t \}.$$

Since each u_k^t is harmonic, we have $\partial^{E_t} * u_k^t \equiv 0$ and

$$(3.124) \quad \partial^{E_t} \bar{\partial}^t * u_k^t = \Theta(E_t, h) * u_k^t.$$

Thus (3.122) follows from (3.123) and (3.124). The proof is complete.

4. CURVATURE OF THE WEIL-PETERSSON METRIC

We shall prove Theorem 1.2 in this section. Let $\pi : \mathcal{X} \rightarrow B$ be a proper holomorphic submersion. Then we have the classical Kodaira-Spencer map

$$(4.1) \quad \kappa : v \mapsto \kappa(v) \in H^{0,1}(T_{X_t}) \simeq H^{n,n-1}(T_{X_t}^*)^*, \quad \forall v \in T_t B, \quad t \in B.$$

we shall prove that:

Proposition 4.1. *Assume that the dimension of $H^{0,1}(T_{X_t})$ does not depend on t . Then*

$$(4.2) \quad \kappa : v \mapsto \kappa(v) \in H^{n,n-1}(T_{X_t}^*)^*, \quad \forall v \in T_t B,$$

defines a holomorphic bundle map from T_B to the dual bundle of

$$(4.3) \quad \mathcal{H}^{n,n-1} := \{ H^{n,n-1}(T_{X_t}^*) \}_{t \in B}.$$

Proof. Let $v : t \mapsto v^t$ be a holomorphic vector field on B . Let V be a smooth $(1,0)$ -vector field on the total space such that $\pi_* V = v$. Then we know that for each t , $(\bar{\partial} V)|_{X_t}$ defines a representative of $\kappa(v^t)$. It suffices to prove that

$$(4.4) \quad \pi_*(\mathbf{u} \wedge \bar{\partial} V)$$

is holomorphic for every holomorphic section u of $\mathcal{H}^{n,n-1}$, where \mathbf{u} is an arbitrary representative of u . Let us write

$$(4.5) \quad \bar{\partial}\mathbf{u} = \sum dt^j \wedge a_j + \sum d\bar{t}^k \wedge b_k.$$

By the proof of Theorem 3.1, we know that each $b_k|_{X_t}$ is $\bar{\partial}^t$ -closed and the cohomology class of $b_k|_{X_t}$ does not depend of the choice of b_k . Moreover, if u is a holomorphic section then the cohomology class of $b_k|_{X_t}$ is zero. Thus

$$(4.6) \quad \pi_*(b_k \wedge \bar{\partial}V) = 0,$$

and

$$(4.7) \quad \bar{\partial}\pi_*(\mathbf{u} \wedge \bar{\partial}V) = \pi_*(\bar{\partial}\mathbf{u} \wedge \bar{\partial}V) = \sum d\bar{t}^k \wedge \pi_*(b_k \wedge \bar{\partial}V) = 0,$$

which implies that $\pi_*(\mathbf{u} \wedge \bar{\partial}V)$ is holomorphic. The proof is complete. \square

Proof of Theorem 1.2: Let $v : t \mapsto v^t$ be an arbitrary holomorphic vector field on B . Let us denote by $\|v\|_{WP}$ the upper semicontinuous regularization of

$$(4.8) \quad t \mapsto \|v^t\|_{WP}.$$

Then it suffices to prove that $\log \|v\|_{WP}$ is plurisubharmonic or equal to $-\infty$ identically. Since there is a Zariski open subset, say B_0 , of B such that the dimension of $H^{n,n-1}(T_{X_t}^*)$ is a constant on B_0 , by Theorem 1.1, we know that $\mathcal{H}^{n,n-1}$ is Griffiths-semipositive on the complement of a proper analytic subset. Thus by Proposition 4.1, we know that $\log \|v\|_{WP}$ is plurisubharmonic or equal to $-\infty$ identically on the complement of a proper analytic subset. Now it is sufficient to prove that $\|v\|_{WP}$ is locally bounded from above.

Let V be a smooth $(1,0)$ -vector field on the total space such that $\pi_*V = v$. By definition, we know that

$$(4.9) \quad \|v^t\|_{WP} \leq \|(\bar{\partial}V)|_{X_t}\|_{H^{0,1}(T_{X_t})}.$$

Since

$$(4.10) \quad t \mapsto \|(\bar{\partial}V)|_{X_t}\|_{H^{0,1}(T_{X_t})},$$

is smooth, we know that $\|v\|_{WP}$ is locally bounded from above. The proof is complete.

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